On the Decay of Symmetric Toroidal Dynamo Fields

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To Professor Arnulf Schlüter on his 60th Birthday

The "anti-dynamo" theorems for toroidal magnetic fields with axisymmetry and plane symmetry are generalized to the case of a compressible, time-dependent flow in a fluid with arbitrary conductivity.

1. Introduction

"Anti-dynamo" theorems (see for instance [1—5]) state that self-sustained dynamo action in an electrically conducting fluid is impossible if both the electromagnetic field and the fluid velocity possess certain symmetries. For the case of axisymmetry and plane symmetry it has been shown in [5, 6] that the externally visible poloidal part of the field decreases monotonically in time, even if the flow is compressible and time-dependent. Here, we construct a Liapunov function which shows that the toroidal field cannot grow in time. This, together with the results in [5, 6], refutes speculations in the recent literature [7] that the axisymmetric electromagnetic field might possibly grow if the flow is both compressible and time-dependent.

2. Plane Symmetry

Using Cartesian coordinates and the notation of [6], we introduce poloidal and toroidal functions g and h by

$$B_x = (\partial g/\partial y), \ B_y = - (\partial g/\partial x), \ B_z = h.$$

In [6] it was shown that the poloidal function g decays. If there is no poloidal field, the toroidal function h has to satisfy

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(\eta \frac{\partial}{\partial x} h - v_x h\right) \tag{1}$$
$$+ \frac{\partial}{\partial y}\left(\eta \frac{\partial}{\partial y} h - v_y h\right) \text{ in } G \times (0, \infty)$$

with the initial-boundary conditions

$$h(\boldsymbol{x}, t) = 0 \quad \text{for} \quad \boldsymbol{x} \in \partial G$$
$$\text{and all} \quad t > 0, \tag{2a}$$
$$h(\boldsymbol{x}, 0) = h_0(\boldsymbol{x}), \quad \boldsymbol{x} = (x, y) \in G. \tag{2b}$$

The impossibility of dynamo action now follows from:

Theorem 1: Let $G \subset \mathbb{R}^2$ be a bounded simply connected domain. Let the boundary ∂G be $C^{1+\alpha}$-smooth. Furthermore, we assume that η, v_x, v_y, and h_0 have Hölder continuous derivatives in $\bar{G} \times [0 \ \infty)$ with $\eta(\boldsymbol{x}, t) \geqq \eta_0 > 0$, and on $\partial G \ \boldsymbol{v} \cdot \boldsymbol{n} = 0$, $\boldsymbol{n}$ being the outer normal at ∂G. Then if h is a solution of (1) and (2),

$$\langle |h| \rangle := \int\limits_G |h(x, y, t)| \, \mathrm{d}x \, \mathrm{d}y \leq H(t), \tag{3}$$

and $H(t)$ decays for all times.

Note that the r.h.s. of (1) contains the term $h \operatorname{div} \boldsymbol{v}$. $\operatorname{div} \boldsymbol{v} \neq 0$ means compressibility of the fluid and $\operatorname{div} \boldsymbol{v}$ always changes sign in G because of $\langle \operatorname{div} \boldsymbol{v} \rangle = 0$. Thus neither the maximum principles of [8] nor the known theorems on the asymptotic decay of solutions (see [9] p. 158) are applicable. Also, the approach of [10] for compressible time-independent flow cannot easily be generalized to this case. However, our proof of Theorem 1 is closely related to the method used by Braginskii [2] (compare Section 3).

Proof: We use the following abbreviations:

$$\partial_j v_j := \operatorname{div} \boldsymbol{v} = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y,$$

$$\mathrm{L}u := \partial_j(\eta \, \partial_j u - v_j u).$$

The idea of the proof is as follows: We show that

there are functions $f > 0$ and w such that

$$\langle |h| \rangle^2 \leqq \langle f \rangle \langle f w^2 \rangle , \quad \langle f \rangle = 1$$

and $\langle f w^2 \rangle$ decaying in time.

Let f be the solution of

$$\dot{f} = L f \quad \text{in} \quad G \times \mathbb{R}_+ , \tag{4a}$$

$$\partial f / \partial n (\boldsymbol{x}, t) = 0 \quad \text{for} \quad \boldsymbol{x} \in \partial G , \quad t > 0 , \tag{4b}$$

$$f(\boldsymbol{x}, 0) = 1 / \langle 1 \rangle \quad \text{for} \quad \boldsymbol{x} \in \bar{G} . \tag{4c}$$

Our smoothness assumptions are such that the existence of f follows from [9] p. 147. By elementary methods and maximum principle techniques ([8], Theorem 7, p. 174f), it can be shown that $f(\boldsymbol{x}, t) > 0$ in $\bar{G} \times [0, T]$ for any $T > 0$, and that f grows or decays at most exponentially, the rate depending on the bound for $|\mathrm{div}\, \boldsymbol{v}|$. Thus $f > 0$ exists for all times and the function w defined by $w(\boldsymbol{x}, t) = h(\boldsymbol{x}, t) / f(\boldsymbol{x}, t)$ in $\bar{G} \times \mathbb{R}_+$ also exists.

We now show that

$$\partial / \partial t \langle f w^2 \rangle = - \langle 2 \eta f |\nabla w|^2 \rangle \leqq 0 . \tag{5}$$

For every fixed time $t > 0$ we get by partial integration

$$\langle w L h \rangle = \langle w \, \partial_j (\eta \, \partial_j f \, w - v_j f w) \rangle$$
$$= - \langle (\partial_j w) [\eta f \, \partial_j w + \eta w \, \partial_j f - v_j f w] \rangle$$
$$= - \langle \eta f |\nabla w|^2 \rangle - \left\langle \left(\partial_j \frac{w^2}{2} \right) (\eta \, \partial_j f - v_j f) \right\rangle$$
$$= - \langle \eta f |\nabla w|^2 \rangle + \left\langle \frac{w^2}{2} L f \right\rangle . \tag{6}$$

In these equalities, we skipped terms that can be written as an integral over ∂G and vanish because $h = f w$ and $\partial f / \partial n$ vanish for $\boldsymbol{x} \in \partial G$. We also get

$$\langle w \dot{h} \rangle = \langle w (\dot{f} w + \dot{w} f) \rangle$$
$$= \left\langle \frac{w^2}{2} \right\rangle^{\cdot} + \left\langle \dot{f} \frac{w^2}{2} \right\rangle . \tag{7}$$

Together with (1), (2), and (4), the subtraction of (7) from (6) now yields (5). We apply the Schwarz inequality and get

$$\langle |h| \rangle^2 = \langle f |w| \rangle^2 = \langle f^{1/2} \cdot f^{1/2} |w| \rangle^2$$
$$\leqq \langle f \rangle \cdot \langle f w^2 \rangle .$$

According to (5), the Liapunov function $H^2(t) := \langle f w^2 \rangle$ decays in time. The proof is thus completed after we have shown that $\langle f \rangle$ does not grow in time. This follows immediately from (4a), (4b):

$$\langle f \rangle^{\cdot} = \langle \dot{f} \rangle = \langle \partial_j (\eta \, \partial_j f - v_j f) \rangle$$
$$= \int_{\partial G} (\eta \, \partial f / \partial n - f \boldsymbol{v} \cdot \boldsymbol{n}) \, d\tau = 0 ,$$

and this means $\langle f \rangle = $ const. From the initial condition (4c) now follows $\langle f \rangle = 1$.

3. Axisymmetry

The possibility of an axisymmetric dynamo in a 3-dimensional, axisymmetric domain G with smooth boundary ∂G is considered. We introduce cylindrical coordinates r, φ, z and use the same notations as in [6]. The magnetic field $\boldsymbol{B}$ is then given by

$$B_r = - \frac{1}{r} \frac{\partial \psi}{\partial z} , \quad B_\varphi = q , \quad B_z = \frac{1}{r} \frac{\partial \psi}{\partial r} .$$

It is assumed to be a differentiable, axisymmetric vector field. If there is no poloidal field ($\psi = 0$), the toroidal part q of the field must satisfy

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial r} \frac{\eta}{r} \frac{\partial}{\partial r} r q + \frac{\partial}{\partial z} \eta \frac{\partial}{\partial z} q$$
$$- \frac{\partial}{\partial r} q v_r - \frac{\partial}{\partial z} q v_z \quad \text{in} \quad G , \tag{8a}$$

$$q(r, z, t) = 0 \quad \text{on} \quad \partial G \quad \text{for all} \quad t \geqq 0 , \tag{8b}$$

$$q(r, z, 0) = q_0(r, z) \tag{8c}$$

with a q_0 such that (8b) and (8d) below are satisfied for $t = 0$.

Condition (8c) excludes artificial singularities caused by improper initial values. Moreover, the current density must always stay finite in G. This means that B_φ / r must be bounded and thus follows from the requirement that $\boldsymbol{B}$ be differentiable in G. This leads to the condition

$$\lim_{\substack{r \to 0 \\ (r, z) \in G}} \frac{|q(r, z, t)|}{r} \leqq c < \infty \quad \text{for all} \quad t \geqq 0 , \tag{8d}$$

in the case that G contains the axis $r = 0$. We introduce the function $p := q / r$ and require that it be a classical solution of the reformulated Eqs. (9a—c):

$$\dot{p} = \mathrm{div} \, [\eta / r^2 \, \nabla r^2 \, p - p \boldsymbol{v}] , \tag{9a}$$

$$p(r, z, t) = 0 \quad \text{on} \quad \partial G \quad \text{for all} \quad t \geqq 0 , \tag{9b}$$

$$p(r, z, 0) = p_0(r, z) \quad \text{with a } p_0 \text{ that vanishes}$$
$$\text{on } \partial G . \tag{9c}$$

Here $\mathrm{div} \, \boldsymbol{u} = (1/r) \, \partial_r r u_r + \partial_z u_z$ denotes the 3-dimensional divergence of the axisymmetric vector fiield $\boldsymbol{u}$. We now have the following theorem:

Theorem 2: Let G be a bounded, axisymmetric domain in $\mathbb{R}^3$, topologically equivalent to a ball or a torus. Let its boundary ∂G be smooth ($C^{1+\alpha}$). Furthermore, let us assume that η, $\boldsymbol{v}$ and q_0 have

continuous second derivatives in $\bar{G} \times \mathbb{R}_+$, $\eta(r, z, t) \geqq \eta_0 > 0$ and on ∂G $\boldsymbol{v} \cdot \boldsymbol{n} = 0$, $\boldsymbol{n}$ being the outer normal at ∂G. Then if p is a solution of (9),

$$\langle |p| \rangle_G := 2\pi \int\limits_G |p(r, z, t)| \, r \, \mathrm{d}r \, \mathrm{d}z \leqq P(t)$$

and $P(t)$ decays for all times.

Remark 1. The assumptions of Theorem 2 have been chosen for our convenience and could certainly be relaxed in several respects.

Remark 2. The existence of the second derivatives of $\boldsymbol{v}$ and η ensures that $\dfrac{\partial_r \eta(r, z, t)}{r}$ and $\dfrac{v_r(r, z, t)}{r}$ stay bounded in the limit $r \to 0$.

Proof: The idea of the proof is the same as for Theorem 1. If G contains the axis $r = 0$, however,

the singularity of the coefficients in (9a) requires some special care. We thus perform here the proof for this case, the other one being similar, but easier. For any given $\varepsilon > 0$ we define

$$G_\varepsilon := \{(r, z) \in G : r > \varepsilon\} \quad \text{with boundary } \partial G_\varepsilon.$$

Put $p = fw$, where f is the solution of

$$\dot{f} = \operatorname{div}\left(\frac{\eta}{r^2} \nabla r^2 f - f\boldsymbol{v}\right) \text{in } G, \tag{10a}$$

$$\partial_n(r^2 f) = 0 \quad \text{on} \quad \partial G \times \mathbb{R}_+, \tag{10b}$$

$$f(r, z, 0) = f_0(r, z) > 0, \tag{10c}$$

and $f_0(r, z)$ is a two times differentiable function in $\bar{G}$ satisfying $\partial_n(r f_0) = 0$ on ∂G. We shall show later that $f \geqq 0$ in $\bar{G} \times \mathbb{R}_+$. By partial integration we get

$$
\begin{aligned}
\langle w(\dot{f}w + \dot{w}f) \rangle_{G_\varepsilon} &= \left\langle \dot{f}\frac{w^2}{2} \right\rangle_{G_\varepsilon} + \left\langle \frac{w^2}{2}\dot{f} \right\rangle_{G_\varepsilon} = \int\limits_{\partial G_\varepsilon} \left[w\frac{\eta}{r^2}(w\,\partial_n r^2 f + r^2 f\,\partial_n w) - w^2 f v_n \right] \mathrm{d}^2 S \\
&\quad - \left\langle (\nabla w) \cdot \left[\frac{\eta}{r^2}(w\,\nabla r^2 f + r^2 f\,\nabla w) - fw\boldsymbol{v} \right] \right\rangle_{G_\varepsilon} \\
&= \int \left[2\pi r\left(w\frac{\eta}{r^2}(-w\,\partial_r r^2 f - r^2 f\,\partial_r w) + w^2 f v_r \right) \right]_{r=\varepsilon} \mathrm{d}z \\
&\quad - \left\langle \left(\nabla\frac{w^2}{2} \right)\left(\frac{\eta}{r^2}\nabla r^2 f - f\boldsymbol{v} \right) \right\rangle - \langle \eta f |\nabla w|^2 \rangle \\
&= -\int \left[2\pi r\left(w^2\frac{\eta}{r^2}\partial_r r^2 f + w\eta f\,\partial_r w - w^2 f v_r \right) \right]_{r=\varepsilon} \mathrm{d}z \\
&\quad - \int\limits_{\partial G_\varepsilon} \frac{w^2}{2}\left(\frac{\eta}{r^2}\partial_n r^2 f - f v_n \right) \mathrm{d}^2 S + \left\langle \frac{w^2}{2}\operatorname{div}\left(\frac{\eta}{r^2}\nabla r^2 f - f\boldsymbol{v} \right) \right\rangle - \langle \eta f |\nabla w|^2 \rangle \\
&= -\int \left[\pi r w^2 \frac{\eta}{r^2}f\,\partial_r r^2 \right]_{r=\varepsilon} \mathrm{d}z + O(\varepsilon) + \left\langle \frac{w^2}{2}\dot{f} \right\rangle_{G_\varepsilon} - \langle \eta f |\nabla w|^2 \rangle_{G_\varepsilon} \\
&= -\int [2\pi w^2 \eta f]_{r=\varepsilon} \mathrm{d}z - \langle \eta f |\nabla w|^2 \rangle_{G_\varepsilon} + O(\varepsilon) + \left\langle \frac{w^2}{2}\dot{f} \right\rangle_{G_\varepsilon}.
\end{aligned}
$$

In the limit $\varepsilon \to 0$, this gives

$$\left\langle \dot{f}\frac{w^2}{2} \right\rangle_G = -\langle \eta f |\nabla w|^2 \rangle_G - \int\limits_{I(G)} [2\pi w^2 \eta f]_{r=0}\,\mathrm{d}z \leqq 0, \tag{11}$$

where $I(G)$ denotes the z-interval given by the intersection of G with the axis $r = 0$. In the computations leading to (11), it was concluded that

$$\int\limits_{\partial G_\varepsilon \cap \partial G} \left[w\frac{\eta}{r^2}(w\,\partial_n r^2 f + r^2 f\,\partial_n w) - w^2 f v_n \right] \mathrm{d}^2 S = 0$$

from $\partial_n r^2 f = 0$ and $fw = p = 0$ on ∂G. In the computations of

$$\int\limits_{\partial G_\varepsilon \cap \partial G} [\ldots] \mathrm{d}^2 S$$

we made use of Remark 2. In a very similar way we also get

$$\langle f \rangle_G^{\cdot} = -\int\limits_{I(G)} [4\pi \eta f]_{r=0}\,\mathrm{d}z \leqq 0, \tag{12}$$

but this time we need $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on ∂G. From the Cauchy-Schwarz inequality it follows that

$$
\begin{aligned}
\langle |p| \rangle^2 &= \langle \sqrt{f} \cdot \sqrt{f} \, |w| \rangle^2 \\
&\leqq \langle f \rangle \langle f w^2 \rangle =: P^2(t),
\end{aligned}
$$

and $P(t)$ decreases in time. The proof of Theorem 2 is thus completed once it has been shown that $f(r, z, t) \geqq 0$ in $\bar{G} \times \mathbb{R}_+$.

Remark 3: If $\eta = \text{const}$ and $\text{div } v = 0$, $f(r, z, t) \equiv 1$ satisfies (10a) and $\langle f \rangle^{\cdot} = 0$. It also allows (11) to be deduced and is thus a possible choice for solenoidal flows. In this case, our proof reduces to the one given by Braginskiĭ [2]. We note, however, that the term corresponding to the contribution of the surface integral in (11) is missing in (2.9b) of [2].

It is now shown that $f(r, z, t) \geqq 0$ in $\bar{G} \times \mathbb{R}_+$. Following [1], we reformulate (10) in the axisymmetric domain $G_5 \subset \mathbb{R}^5$, using the relations

$$r = \sqrt{x_i x_i} = \sqrt{x_1^2 + \cdots + x_4^2}$$

and

$$\tilde{f}(x_1, \ldots, x_4, z, t) = f(\sqrt{x_i x_i}, z, t),$$
$$i = 1, \ldots, 4:$$

$$\dot{\tilde{f}} = \eta \Delta_5 \tilde{f} - u_\nu \partial_\nu \tilde{f} + a \tilde{f} \quad \text{in } G_5,$$
$$\nu = 1, \ldots, 5; \tag{13a}$$

$$\partial_n (x_i x_i \tilde{f}) = 0 \quad \text{on} \quad \partial G_5 \times \mathbb{R}_+, \tag{13b}$$

$$\tilde{f}(x_1, \ldots, x_4, z, 0) = f_0(\sqrt{x_i x_i}, z),$$
$$i = 1, \ldots, 4. \tag{13c}$$

As will be shown in the Appendix, (13a) is a uniformly parabolic equation with bounded coefficients. We chose λ such that $a(\sqrt{x_i x_i}, z, t) - \lambda \leqq 0$ and apply the maximum principle Theorem 7 of [8, p.174f] to $g = -e^{-\lambda t} \tilde{f}$. It then follows that $g(\sqrt{x_i x_i}, z, t) < 0$ in $G_5 \times [0, T]$ for any $T > 0$, and thus

$$f(\sqrt{x_i x_i}, z, t) \geqq 0 \quad \text{in} \quad \bar{G} \times \mathbb{R}_+.$$

Appendix

In [1], an argument simplified considerably because Backus/Chandrasekhar embedded their 3-dimensional axisymmetric problem into the (physically meaningless) $\mathbb{R}^5$. They assumed

$$r = \sqrt{x_1^2 + \cdots + x_4^2}$$

and thus got

$$r^2 \Delta_5 f = \Delta^* r^2 f. \tag{A1}$$

Here $f = f(r, z)$ is an axisymmetric function, Δ_5 is the Laplacian in $\mathbb{R}^5$, and

$$\Delta^* \cdot = \partial_{rr} \cdot - \frac{1}{r} \partial_r \cdot + \partial_{zz} \cdot.$$

For verification of (A1), and also for the computations given below, it is useful to know the following formula in $\mathbb{R}^n$, $n \geqq 2$:

$$\Delta_n f = f_{rr} + \frac{n-2}{r} f_r + f_{zz}, \quad r = \sqrt{x_i x_i},$$
$$i = 1, \ldots, n-1. \tag{A2}$$

We now show how (10) can be transformed into (13). We rewrite (10a) as

$$\begin{aligned}
\dot{f} &= \frac{1}{r} \partial_r \left(r \frac{\eta}{r^2} \partial_r r^2 f - r f v_r \right) + \partial_z \eta \, \partial_z f - \partial_z f \, v_z \\
&= \frac{1}{r} (2 \eta \, \partial_r f + 2 f \, \partial_r \eta + \eta \, \partial_r f + r \, \partial_r \eta \, \partial_r f \\
&\quad - \partial_r f \, r \, v_r) + \partial_z \eta \, \partial_z f - \partial_z f \, v_z \\
&= \eta \left(\partial_{rr} f + \partial_{zz} f + \frac{3}{r} \partial_r f \right) \\
&\quad - (v_r - \partial_r \eta) \, \partial_r f - (v_z - \partial_z \eta) \, \partial_z f \\
&\quad + \left(\frac{2}{r} \partial_r \eta - \frac{1}{r} \partial_r r \, v_r - \partial_z v_z \right) f.
\end{aligned}$$

With

$$r = \sqrt{x_i x_i}, \quad i = 1, \ldots, 4, \quad \tilde{f}(x_1, \ldots, x_4, z)$$
$$= f(\sqrt{x_i x_i}, z), \frac{\partial \tilde{f}}{\partial x_i} = \frac{x_i}{r} \partial_r f,$$

Eq. (A2) and

$$x u_i(x_1, \ldots, x_4, z) := (x_i/r)(v_r - \partial_r \eta),$$
$$i = 1, \ldots, 4,$$
$$u_5(x_1, \ldots, x_4, z) := v_z - \partial_z \eta,$$
$$a(x_1, \ldots, x_4, z) := \frac{2}{r} \partial_r \eta - \frac{1}{r} \partial_r r \, v_r - \partial_z v_z$$

we now get

$$\dot{\tilde{f}} = \eta \Delta_5 \tilde{f} - u_\nu \partial_\nu \tilde{f} + a \tilde{f}, \quad \nu = 1, \ldots, 5,$$

and this is (13a). The boundedness of a and u_ν follows from our smoothness assumptions and Remark 2.

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